See email for exam 2 stats (well done!).

Closing *Thurs*: 4.4

Closing next *Tues*: 4.4-5

Closing next *Thurs*: 4.7 (last assignment)

Entry Task: **Review**! How would you evaluate these old final questions:

1. $\lim_{x\to 0} \frac{e^x - x}{5\cos(x) + 3\sin(x)}$

2. $\lim_{x\to 1^+} \frac{x-10}{x(1-x)}$

3. $\lim_{x\to 4} \frac{16-x^2}{4-x}$

4. $\lim_{x\to 0} \frac{\sin(x)}{x}$

5. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$

4.4 L'Hopital's Rule

A *shortcut* method for some limits.

L'Hopital's Rule (0/0 case)

Suppose $\boldsymbol{g(a) = 0}$ **and** $\boldsymbol{f(a) = 0}$

and f and g are differentiable at $x = a$,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Examples:

1. $\displaystyle\lim_{x \to 4} \frac{16 - x^2}{4 - x}$

2. $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x}$

3. $\displaystyle\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$

Aside: Sketch of derivation

Assume $g(a) = 0$ and $f(a) = 0$

(These explanations are for the case when g'(a) is not zero).

Explanation 1 (def'n of derivative)

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

provided these limits exist we have:

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Explanation 2 (tangent line approx.):

The tangent lines for $f(x)$ and $g(x)$ at $x = a$ are

$$y = f'(a)(x-a) + 0$$

$$y = g'(a)(x-a) + 0$$

And we know these approximate the functions f(x) and g(x) better and better the closer x gets to a, so

Thus,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

Sometimes you have to use it more than once.

Example:

$$\lim_{x \to 1} \frac{x - \sin(x - 1) - 1}{(x - 1)^3}$$

L'Hopitals rule can also be used directly for the ∞/∞ case

2. $\lim_{x \to \infty} \frac{\ln(x)}{x}$

Example:

1. $\lim_{x \to \infty} \frac{5x + 7}{6 + 13x}$

3. $\lim_{x \to \infty} xe^{-3x}$

4. $\lim_{x \to \infty} \frac{3\text{x} + 1}{\sqrt{9 + 4x^2}}$

Other indeterminant forms:

0·∞: (rewrite as a fraction)

$$\lim_{x \to 0^+} x \ln(x)$$

$$\lim_{x \to 0^+} x\, \mathrm{e}^{1/\mathrm{x}}$$

$\infty - \infty$: (combine into a fraction)

$$\lim_{t\to\infty} \frac{2}{t(1+3t)^2} - \frac{2}{t}$$

$\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^{\infty}$ **:** (Use ln( ))

$$\lim_{x\to 0^+} x^x$$

$$\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{x}$$

Aside (you don't need to know this): This is an important application of what we just discussed:

The formula for compound interest is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

P = starting balance,
A = end balance,
r = annual rate,
n = number of times interest is compounded each year,
t = number of years

In some bank accounts interest is computed once a month, for some every day, for some every second. If you wanted interested to always be computed (continuously), then the new formula would be

$$A = \lim_{n \to \infty} P\left(1 + \frac{r}{n}\right)^{nt}$$

You can use the techniques just discussed to find this limit and you get $\lim_{n \to \infty} P\left(1 + \frac{r}{n}\right)^{nt} = Pe^{rt}$

Thus,

$$A = Pe^{rt}$$

is the *continuous compounding* interest formula.